Wreath products, cofinitely equivariant maps, and orbit equivalence

joint with Robin Tucker-Drob

Γ cntble discrete $\quad \Gamma \curvearrowright (X,\mu)$ free pmp ergodic

Def $\Gamma \curvearrowright (X,\mu) \sim_{OE} \Lambda \curvearrowright (Y,\nu)$
if $\exists$ measure isom. $\phi : (X,\mu) \to (Y,\nu)$
s.t. $\phi(\Gamma \cdot x) = \Lambda \cdot \phi(x)$

Def $\Gamma \curvearrowright (X,\mu) \sim_{SOE} \Lambda \curvearrowright (Y,\nu)$
if $\exists$ pos. meas. $X' \subseteq X$, $Y' \subseteq Y$
and a meas. isom $\phi : (X', \mu_{X'}) \to (Y', \nu_{Y'})$
st. $\phi(\Gamma_x \cap X') = \Lambda \cdot \phi(x) \cap Y'$

Def $\Gamma \sim_{OE} \Lambda := \exists$ free pmp ergodic $\Gamma \curvearrowright (X,\mu) \sim_{OE} \Lambda \curvearrowright (Y,\nu)$
similarly for SOE

Ex if A, B are finite then $A \sim_{SOE} B$
$|A| = |B| \iff A \sim_{OE} B$

Thm (Ornstein-Weiss) Any pair of free pmp ergodic actions of inf. amenable groups is orbit equivalent.

Wreath products $\quad A \wr \Gamma := \bigoplus_{\Gamma} A \rtimes \Gamma$

• (Open) $L(\Gamma) \cong L(\Lambda) \Rightarrow \Gamma \sim_{SOE} \Lambda$?

Thm (Bowen) $L(A \wr \mathbb{F}_2) \cong L(B \wr \mathbb{F}_2)$
if A, B nontrivial amenable.

Prop If $\Lambda_1 \sim_{OE} \Lambda_2$, then $\Lambda_1 \wr \Gamma \sim_{OE} \Lambda_2 \wr \Gamma$
for every countable discrete Γ.

Pf $\Lambda_1 \curvearrowright (X,\mu) \sim_{OE} \Lambda_2 \curvearrowright (Y,\nu)$

$$\underset{\bigoplus_{\Gamma} \Lambda_1 \rtimes \Gamma}{\Lambda_1 \wr \Gamma} \curvearrowright (X^{\Gamma}, \mu^{\Gamma}) \qquad \big(\underset{f \in \Lambda}{(\lambda}, \gamma) \cdot f\big)(\omega) = \overset{\in \Lambda_1}{\lambda(\gamma^{-1}\omega)} \cdot \overset{\in X}{f(\gamma^{-1}\omega)}$$

$$\Lambda_2 \wr \Gamma \curvearrowright (Y^{\Gamma}, \nu^{\Gamma})$$ $\square$

$$\Phi : X^{\Gamma} \to Y^{\Gamma}$$
$$\Phi(f)(\gamma) = \phi(f(\gamma))$$

What about $\Lambda_1 \sim_{SOE} \Lambda_2$? $\qquad R_{A \times \mathbb{Z}} \cong R_{A \times \mathbb{Z}} \times R_{\mathbb{Z}}$

$C_2 \sim_{SOE} C_3$

Thm (Tucker-Drob-W) The wreath product actions of $A \wr \mathbb{F}_2$ and $B \wr \mathbb{F}_2$ are orbit equivalent for any nontrivial amenable A, B.

Cor • $\Lambda \wr \mathbb{F}_2 \sim_{OE} (\Lambda \times A) \wr \mathbb{F}_2$ for any nontrivial amenable A and nontrivial Λ.
• $\Lambda_1 \sim_{SOE} \Lambda_2 \Rightarrow \Lambda_1 \wr \mathbb{F}_2 \sim_{OE} \Lambda_2 \wr \mathbb{F}_2$

Pf of 1st point $\quad (\Lambda \times C_2) \wr \mathbb{F}_2 \sim_{OE} (\Lambda \times \mathbb{Z}) \wr \mathbb{F}_2$

$((\lambda \times c), \gamma) \to (\lambda, \gamma) \quad (c, \gamma)$ $\qquad$ apply our theorem

• $(\Lambda \times C_2) \wr \mathbb{F}_2 \sim_{OE} \underbrace{\Lambda \wr \mathbb{F}_2}_{} \times C_2 \wr \mathbb{F}_2 \sim_{OE} \Lambda \wr \mathbb{F}_2 \times \mathbb{Z} \wr \mathbb{F}_2$

$\bigoplus_{\mathbb{F}_2} \Lambda \times C_2 \cong \bigoplus_{\mathbb{F}_2} \Lambda \times \bigoplus_{\mathbb{F}_2} C_2$ $\qquad \sim_{OE} (\Lambda \times \mathbb{Z}) \wr \mathbb{F}_2$ $\square$

Thm (Tucker-Drob-W) Let Γ be sofic property (T) icc group.
Assume A, B amenable and
$\alpha : A \wr \Gamma \curvearrowright (X^{\Gamma}, \mu^{\Gamma})$ $\quad \beta : B \wr \Gamma \curvearrowright (Y^{\Gamma}, \nu^{\Gamma})$
are the wr. prod. actions.
TFAE • $\alpha \sim_{SOE} \beta$
• $\alpha \sim_{OE} \beta$
• $|A| = |B|$

Sketch of Proof

Step 1 $\quad$ Find

$$C_2 \wr \mathbb{Z} \curvearrowright (2^{\mathbb{Z}}, \mu^{\mathbb{Z}}) \xrightarrow{\phi \text{ OE}} C_3 \wr \mathbb{Z} \curvearrowright (3^{\mathbb{Z}}, \nu^{\mathbb{Z}})$$
$$\boxed{\bigoplus_{\mathbb{Z}} C_2} \curvearrowright (2^{\mathbb{Z}}, \mu^{\mathbb{Z}}) \xrightarrow[\phi \text{ OE}]{} \bigoplus_{\mathbb{Z}} C_3 \curvearrowright (3^{\mathbb{Z}}, \nu^{\mathbb{Z}})$$

Consequence of (Golodets-Sinel'shchikov, Feldman-Sutherland-Zimmer)

Step 2 **Def** A map $\phi : 2^{\mathbb{Z}} \to 3^{\mathbb{Z}}$ is cofinitely equivariant
if for $x, y \in 2^{\mathbb{Z}}$
$x \sim y \Rightarrow \phi(\gamma \cdot x) \sim \gamma \cdot \phi(y) \quad \forall \gamma \in \mathbb{Z}$

differ in at most fin. many coordinates.

Step 3 Show that if $\phi : 2^{\mathbb{Z}} \to 3^{\mathbb{Z}}$ cof. eq., then $\exists\, \Phi : 2^{\mathbb{F}_2} \to 3^{\mathbb{F}_2}$ cof. eq. $\square$